

Experimental study of noise-induced phase synchronization in vertical-cavity lasers

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We report the experimental evidence of noise-induced phase synchronization in a vertical-cavity laser. The polarized laser emission is entrained with the input periodic pump modulation when an optimal amount of white, Gaussian noise is applied. We characterize the phenomenon, evaluating the average frequency of the output signal and the diffusion coefficient of the phase difference variable. Their values are roughly independent of the different wave forms of periodic input, provided that a simple condition for the amplitudes is satisfied. The experimental results are compared with numerical simulations of a Langevin model.

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The phenomenon of phase synchronization (PS) has been the subject of extensive investigation during the past years (see, e.g., Ref. [1] for a recent review). In the purely deterministic case, two systems with instantaneous phases $\Phi_1(t)$ and $\Phi_2(t)$ are locked if, for two integers n, m , $n\Phi_1 - m\Phi_2$ remains bounded for all times. In the presence of unbounded fluctuations, this condition on the phase difference cannot hold for all times. Sufficiently large noise will eventually cause phase slippage and PS can thus only be effective. For example, for weak periodic forcing of an autonomous stochastic oscillator, the dynamics of the instantaneous phase difference ϕ is ruled by the Adler equation [2,3]

$$\dot{\phi} = \Delta - \Delta_s \sin \phi + \xi, \quad (1)$$

where ξ is a Gaussian white noise. Equation (1) describes the motion of an overdamped Brownian particle in the tilted potential $-\Delta\phi - \Delta_s \cos \phi$. Therefore, if the frequency mismatch Δ is smaller than the synchronization bandwidth Δ_s , locking can occur. However, from time to time fluctuations suffice to kick ϕ out of the well yielding a phase slip. Recent observations of noise-enhanced PS have been reported in more complex situations, where chaos occurs in the absence of noise [4,5]. In this context, dynamical features may be significantly altered and a richer variety of phenomena may arise [6].

For noise-driven bistable systems, PS is strictly related to the phenomenon of stochastic resonance (SR), i.e., the enhancement in response to a weak signal superimposed on a stochastic input (see, e.g., Ref. [7]). The very fact that noise can induce synchronized jumps between the two stable output states has been recently interpreted as a noise-induced PS. The first observation of PS in an electronic circuit has been reported in Ref. [8], analyzing the output mean frequency as a function of added noise. Numerical simulations of Langevin models [9,10] showed a similar behavior and allowed for a more refined description of PS in terms of the

effective phase diffusion coefficient [3]. An analytic theory, performed in the case of a periodic and aperiodic square-wave modulation, was presented by Freund *et al.* in Ref. [11], where the explicit expressions for the mean output frequency and the effective diffusion coefficient were derived from a master equation for the stochastic variable ϕ .

In this work, we present a detailed experimental investigation of noise-induced PS regimes in a bistable optical system. Our study is motivated by the necessity to perform a thoughtful test of the statistical indicators of PS in reliable, well controllable laboratory conditions. The effect of both sinusoidal and square-wave input periodic modulations are considered; the latter case allows us to discuss a general framework for a comparison with the analytic results of Ref. [11].

Our experimental system is a pump-modulated vertical-cavity surface emitting laser (VCSEL). The VCSEL is a semiconductor laser, with a symmetrical cavity allowing for the emission on two linear, perpendicular polarizations selected by the crystal axis directions (see e.g. [12]). The emission symmetry, usually broken by impurities or optical inhomogeneities leading to a single-polarization emission, may be restored for particular choices of the pump current. In such a case, the system is bistable and exhibits noise-driven, random jumps between the two polarizations ruled by Kramers' law [13]. The experimental evidence of SR [14] and binary aperiodic SR [15] in a VCSEL has been recently reported.

In our setup, we employ a VCSEL lasing at 850 nm, thermally stabilized (better than 1 mK) and with a carefully controlled pump current. The overall stability allows for long-time measurements, even in the presence of critical behaviors. A linear polarization direction in the laser emission is selected using a polarizer and a half-wave plate. The laser intensity is monitored by an avalanche detector and the signal is recorded by a digital scope. An optical isolator prevents from optical feedback effects to occur. The signals from a 10-MHz-bandwidth white-noise generator and a sinusoidal (SIN) or a square-wave (SQR) oscillator are summed and coupled into the laser by means of a bias tee. The period of the input signals was chosen to be $T = 5 \mu\text{s}$ (200 kHz) for

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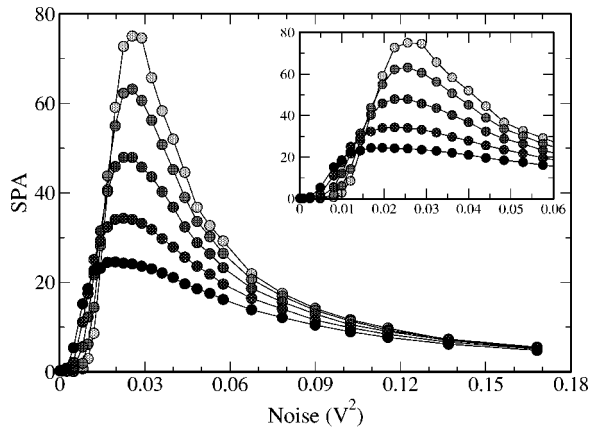


FIG. 1. The spectral power amplification vs applied noise amplitude for different amplitudes of SIN modulation, $A_{SIN} = 50, 100, 150, 200, 250$ mV (top to bottom). The inset is a magnification of the resonance region.

both wave forms. The measurements are performed varying the intensity D of the applied noise as well as the amplitude A of the input modulation. For all the measurements reported henceforth, the A values are subthreshold, i.e., small enough to avoid output transitions every modulation period (in the absence of added noise). The amplitude A_{SQR} is set to be equal to the root mean square value of A_{SIN} , i.e. $A_{SQR} = A_{SIN}/\sqrt{2}$. For every choice of the parameters, a sequence of 16 million points is acquired, corresponding to 500 samples/period for a total of 32 000 periods. Such large datasets allow us to reduce statistical errors to a level of accuracy comparable with current numerical simulations of theoretical models.

To illustrate the phenomenon of SR, we first report in Fig. 1 the response of the system for different values of the amplitude of the SIN modulation, as evaluated by means of the spectral power amplification (SPA) $R_{SPA} = 4 |\mathbf{X}_{out}(\Omega)|^2 / A_{SIN}^2$ [16], where \mathbf{X}_{out} is the Fourier spectrum of the output signal evaluated at the driving frequency $\Omega = 2\pi/T$. A well defined peak of SPA for an optimal value of the applied noise is observed, yielding the typical signature of SR [7]. Increasing the input amplitude, a decrease in

the response as well as a shift towards lower noise values of the peak occurs, as already reported from numerical simulations and observations in nonlinear circuits [17].

The peak in the SPA indicates the improvement of the response of the system at the modulation frequency, thus suggesting that a PS regime is achieved. To investigate such a possibility we define the input-output phase difference $\phi(t) = \Phi_{out}(t) - \Phi_{in}(t)$, where $\Phi_{in}(t) = \Omega t$. The instantaneous phase of a real signal $x(t)$ can be defined in a general way employing the concept of the analytic signal [1]. This is accomplished by considering the complex signal $z(t) = x(t) + iy(t)$, where $y = \mathbf{H}x$ is the usual Hilbert transform of $x(t)$. The phase is then given by the argument of z . For bistable systems an alternative and easier definition can be given based on the knowledge of the sequence of switching times t_n only [1]; Φ_{out} can be defined either as a piecewise-constant function increasing by steps of π at each jump time t_n of the output or by linear interpolation. In the analysis of experimental data we checked that different definitions yield the same results up to statistical accuracy.

In Fig. 2 we report the temporal evolution of ϕ , as obtained from the Hilbert transform method, for both the SIN and SQR wave forms and different values of the added noise. In both cases, a clear signature of PS is found at a well defined (resonant) noise power corresponding to the peak in the SPA. As shown in the figure, for a noise below (above) the resonant value the phase difference drifts towards lower (higher) values. The phase difference at resonance maintains an almost constant value, rarely interrupted by jumps of integer multiples of 2π (phase slips). For comparison, in the insets of Fig. 2 it is shown the temporal behavior of ϕ at resonance for a smaller value of the input signal amplitudes; the phase slips are here more frequent, yielding a faster diffusive motion.

To quantify the observed behaviors, it is useful to introduce the average frequency $\langle \omega_{out} \rangle = \langle \dot{\Phi}_{out} \rangle$ and the effective diffusion coefficient $\mathcal{D}_{eff} = \frac{1}{2} (d/dt) [\langle \phi^2 \rangle - \langle \phi \rangle^2]$ [9]. The measured values of $\langle \omega_{out} \rangle$ and \mathcal{D}_{eff} are reported in Fig. 3 as a function of the noise intensity for different values of the input signal amplitude A . Upon increasing A the region of locking, i.e., the range of noise values for which $\langle \omega_{out} \rangle$

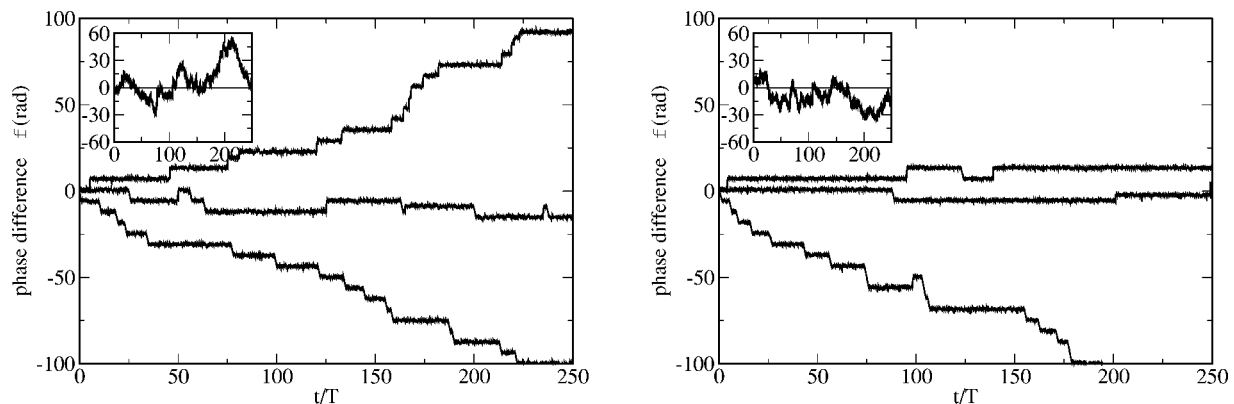


FIG. 2. The phase difference ϕ for SIN (left) and SQR (right) inputs of amplitude $A_{SIN} = 250$ mV. Noise levels are $0.0121 V_{rms}^2, 0.0169 V_{rms}^2, 0.0225 V_{rms}^2$ (bottom to top). The insets are the time series for low-amplitude ($A_{SIN} = 100$ mV) and noise level close to resonance ($0.0196 V_{rms}^2$).

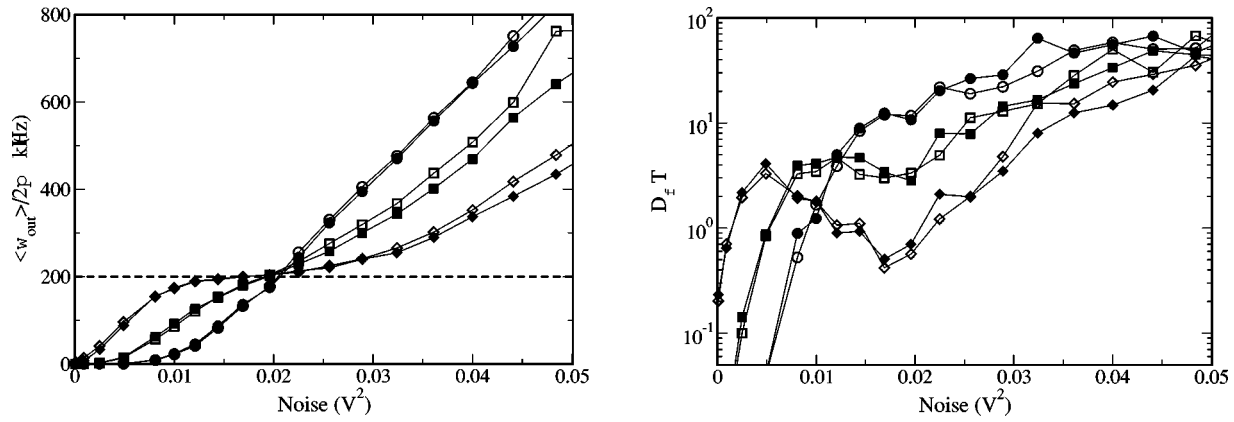


FIG. 3. Average output frequency (left) and diffusion coefficient (right) for sinusoidal and square-wave inputs (full and open symbols, respectively). Input amplitudes are $A_{SIN} = 50, 150,$ and 250 mV (circles, squares, diamonds, respectively).

$\approx \Omega$, widens while D_{eff} develops a pronounced dip at the resonant noise value. It is worth noting that, while the output frequency equals the input frequency at a well defined value $D = D^*$, an increasing A the abscissa of the minimum of D_{eff} shifts towards lower noise amplitudes. Moreover, such abscissa corresponds, within the experimental accuracy, to the abscissa of the maximum of the SPA. Such a result is in agreement with the analytic predictions of Ref. [11].

A related way to illustrate the enhancement of phase coherence is to plot the average duration of locking episodes $\langle T_{lock} \rangle$ (i.e., the average time between consecutive phase slips) as a function of the noise [18]. As seen in Fig. 4, a marked maximum signals strong input-output correlations, namely, a decreasing rate of phase slips (see again Fig. 2). The minimum value of the diffusion constant D_{eff} and the maximal value of $\langle T_{lock} \rangle$ are indicators of the quality of PS. It is therefore important to estimate their dependence on the experimental parameters. On the other hand, the two are not independent: as argued in Ref. [18], the very definition of D_{eff} yields

$$\langle \dot{\phi} \rangle^2 \langle T_{lock} \rangle^2 + 2D_{eff} \langle T_{lock} \rangle = \pi^2. \quad (2)$$

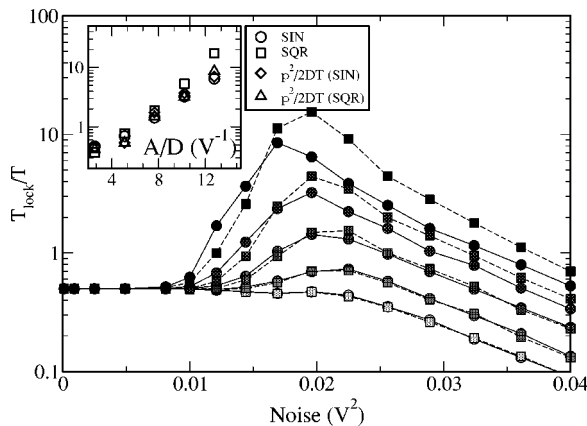


FIG. 4. Average duration of locking episodes for SIN and SQR inputs (circles and squares, respectively), $A_{SIN} = 50, 100, 150, 200,$ and 250 mV (bottom to top). The inset shows $\langle T_{lock} \rangle / T$ and $\pi^2 / 2D_{eff} T$ for $D = D^* = 0.0196V_{rms}^2$ vs A/D (see text).

For $D \approx D^*$, we expect $\langle \dot{\phi} \rangle \approx 0$ and therefore $\pi^2 / 2D_{eff} \approx \langle T_{lock} \rangle$. As shown in the inset of Fig. 4, this estimate is consistent with the experimental data for both input wave forms.

A further relevant information is that $\langle T_{lock} \rangle$ increases exponentially with A/D , confirming that PS rapidly becomes effective upon increasing the driving amplitude. A simple argument to understand this scaling is the following. Consider the case of a two-level system with a slowly modulated barrier $\Delta V(t) = \Delta V \pm B$, where the sign changes every semiperiod $T/2$. The two Kramer's rates $a_{1,2} = r_K \exp(\mp B/D)$ give the probability per unit time to jump from one level to the other. During the locking period, the system stays synchronized with the modulation, likely jumping (almost) every semiperiod. But the probability per unit time for the system to jump less than (or more than) one time per semiperiod (resulting in a phase slip) is approximately a_1 for a sufficiently large B/D . This accounts for the observed scaling if B is proportional to A . The same conclusion can be drawn from Eq. (1). Close to resonance ($\Delta \approx 0$) $D_{eff} \propto \exp(-\text{const } \Delta_s / D)$ [3], and the exponential increase then follows from the fact that $\Delta_s \propto A$ for small amplitudes.

The experimental data are in qualitative agreement with both numerical [9] and analytical [11] studies of noise-induced PS. This originates from the fact that, in the SR regime, the polarization dynamics of our VCSEL can be successfully modeled by the phenomenological Langevin equation [14]

$$\dot{x} = -V'(x) + Af(t) + \eta, \quad (3)$$

where V is a double-well potential, f is the modulation wave form (of frequency Ω), and η is a white, Gaussian noise such that $\langle \eta(t) \eta(t') \rangle = 2D \delta(t - t')$. Here the dimensionless variable x represents the polarized laser intensity. To emphasize the correspondence with the experiment we use here the same symbols A, D, Ω as above. However, one should be aware that this correspondence cannot be established *a priori* since the model is only an effective description. A direct quantitative comparison is feasible only by means of a careful calibration of the physical parameters of model (3) on the

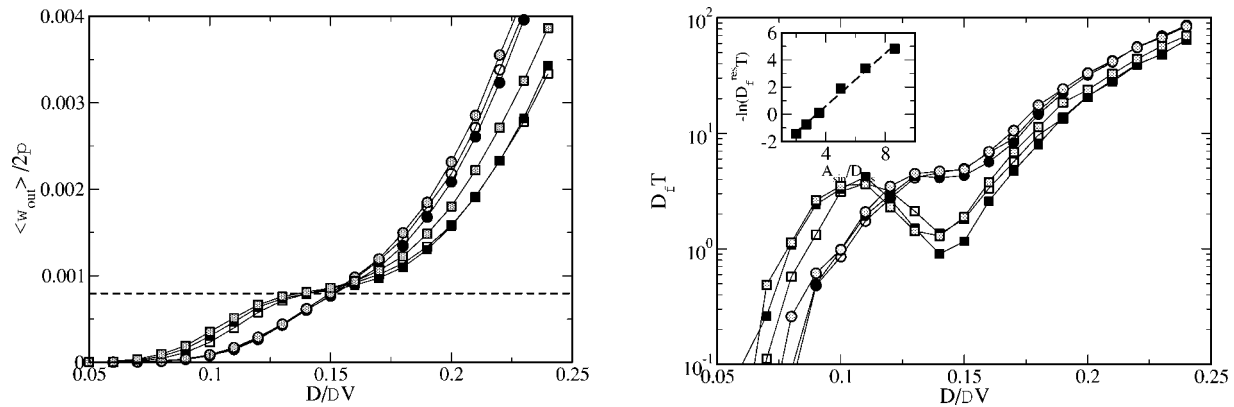


FIG. 5. Simulations of Eq. (3): average output frequency and diffusion coefficient for SIN, SQR, and SAW inputs (full, open, and gray symbols, respectively) with $\Omega = 0.005$ and $A_{\text{SIN}} = 0.3, 0.5$ (circles and squares, respectively).

experimental data. This task has been accomplished in slightly different experimental conditions [14], and goes beyond the scope of the present work. Here, we limit ourselves to illustrate the results of simulations for the model potential $V(x) = x^2(x^2 - 2)$ using a second-order stochastic Runge-Kutta algorithm. The outcomes for moderate values of A ($A \lesssim \Delta V$) shown in Fig. 5 compare well with the experimental ones (see again Fig. 3). Furthermore, the exponential scaling $D_{\text{eff}} \propto \exp(-\text{const } A/D)$ is found, in agreement with experimental findings.

Let us now address the issue of how PS is affected by different choices of driving. Figure 3 indeed shows that both $\langle \omega_{\text{out}} \rangle$ and D_{eff} are approximatively independent of the modulation wave form, provided that $A_{\text{SQR}} = A_{\text{SIN}}/\sqrt{2}$. The only relevant parameter is, at least in this range of amplitudes, the modulation frequency. To check the generality of this observation we performed simulation of Eq. (3) with three different wave forms f , namely, a sinusoidal, a square-wave, and a sawtooth (SAW) with the same rms value, $A_{\text{SQR}} = A_{\text{SIN}}/\sqrt{2} = A_{\text{SAW}}/\sqrt{3}$. As shown in Fig. 5, the curves

are almost overlapped indicating that for moderate amplitudes the quality of PS is largely independent of the details of the driving. To understand this behavior, we compared Kramers' rates for the different wave forms averaged on the modulation semiperiod. Upon changing A/D in the range of interest, we indeed found that they are almost independent of f . Therefore, in the adiabatic limit a comparison with the theory [11] can be worked out and is currently in progress.

In conclusion, we reported a detailed experimental evidence of noise-induced PS in a VCSEL. Phase entrainment is achieved by different input periodic signals, with the same value for the measured statistical indicators as long as the rms of the signal is the same. The numerical analysis of a Langevin model reproduces the features observed experimentally, including the scaling of the locking times at resonance with the amplitude of the modulation.

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